Journal of Engineering Mathematics, Vol. 10, No. 2 April, 1976
Noordhoff International Publishing–Leyden
Printed in The Netherlands

Self-similar behavior of plasma fluid equations, Part II *

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(Received February 25, 1975)

SUMMARY
The "s-parameter transformation group" and "infinitesimal transformation group" are applied to a set of plasma fluid equations to ascertain their self-similar behavior.

1. Introduction

In a previous paper (Hsuan, et al. [1], hereafter labeled I. This paper contains references to previous work.) we examined the self-similar behavior of three sets of fluid equations which are commonly used to describe plasma phenomena. The sets of equations are: (a) multiple species fluid equations truncated at the third moment plus Poisson's equation; (b) massless isothermal *linear* electron fluid and cold nonlinear ion fluid plus Poisson's equation; and (c) massless isothermal electron fluid and cold nonlinear ion fluid with a quasi-neutrality assumption. In this paper, we shall re-examine set (b) with the inclusion of the electron fluid as a massless isothermal *nonlinear* electron fluid.

In Section 2, we shall apply the "s parameter group of transformations" to find the self-similar behavior (Ames [2], Hsuan, et al. [1]). In Section 3, we shall apply the more general "infinitesimal transformation" to ascertain the self similar behavior of this set of equations (Lie [3], Ames [2]). The application of this latter transformation *should* lead to more general invariants with more free parameters which could be later specified to make the equations integrable or fit auxiliary conditions. This has been recently applied successfully to the Korteweg–deVries equation (Shen and Ames [4]).

The set of equations which we wish to examine is:

$$
\begin{aligned}
&\frac{\partial n_i}{\partial t} + \frac{\partial (n_i v_i)}{\partial x} = 0, \\
&\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} = -\frac{\partial \phi}{\partial x}, \\
&\frac{\partial^2 \phi}{\partial x^2} = n_e - n_i, \\
&n_e = \varepsilon^{\phi},
\end{aligned}
\tag{1}
$$

* Supported in part by the National Science Foundation, Grant No. EN6 –7400704.

where all of the symbols are standard. Using this set of equations and with an appropriate rescaling of parameters, it is possible to derive the Korteweg–deVries equation (Washimi and Taniuti [5]). This set of equations has also been extensively investigated numerically to examine, e.g., the evolution of a sheath from a metal boundary whose potential is suddenly decreased (Widner, et al. [6]).

Section 4 is the conclusion and contains a discussion of some solutions of the similar equations.

2. "S-Parameter transformation group"

To find the "s-parameter transformation group" (where $s = 1$), it is advantageous to combine the last of the two equations in (1) such that the set becomes

$$\begin{aligned} &\frac{\partial n_i}{\partial t} + \frac{\partial (n_i v_i)}{\partial x} = 0, \\ &\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} = -\frac{1}{n_e}\frac{\partial n_e}{\partial x}, \\ &\frac{\partial}{\partial x}\left(\frac{1}{n_e}\frac{\partial n_e}{\partial x}\right) = \frac{1}{n_e}\frac{\partial^2 n_e}{\partial x^2} - \left(\frac{1}{n_e}\frac{\partial n_e}{\partial x}\right)^2 = n_e - n_i. \end{aligned} \tag{2}$$

We now follow the procedure given in I and apply the "linear" group defined by

$$G: \begin{cases} \bar{x} = a^{\alpha_1} x, & \bar{t} = a^{\alpha_2} t, \\ \bar{n}_i = a^{\beta_1} n_i, & \bar{v}_i = a^{\beta_2} v_i, \quad \bar{n}_e = a^{\beta_3} n_e \end{cases} \tag{3}$$

to the set given in (2). The α_i's and β_j's are determined such that the set of equation is "(absolutely) constant conformally invariant" under the group G (Ames [2]). A function $F_j(x_i)$ is said to be "constant conformally invariant" (CCI) under G if $F_j(x_i) = f_j(a) F_j(\bar{x}_i)$ where f_j is some function of the parameter a. If $f_j(a) = 1$, then the constant conformal invariance is called "absolutely" (ACCI). The requirement that equations (2) be CCI under G is satisfied if

$$\alpha_1 = \alpha_2, \quad \beta_1 = \beta_3 = -2\alpha_2, \quad \beta_2 = 0. \tag{4}$$

The invariants of the transformation group G are obtained by employing a theorem from group theory (Ames [2]) i.e., $QI \equiv 0$ where I is an invariant and Q is the operator

$$\begin{aligned} Q \equiv &\left.\frac{\partial \bar{x}}{\partial a}\right|_{a=1} \frac{\partial}{\partial x} + \left.\frac{\partial \bar{t}}{\partial a}\right|_{a=1} \frac{\partial}{\partial t} + \left.\frac{\partial \bar{n}_i}{\partial a}\right|_{a=1} \frac{\partial}{\partial n_i} \\ &+ \left.\frac{\partial \bar{v}_i}{\partial a}\right|_{a=1} \frac{\partial}{\partial v_i} + \left.\frac{\partial \bar{n}_e}{\partial a}\right|_{a=1} \frac{\partial}{\partial n_e} \\ = &\ \alpha_1 x \frac{\partial}{\partial x} + \alpha_2 t \frac{\partial}{\partial t} + \beta_1 n_i \frac{\partial}{\partial n_i} + \beta_2 v_i \frac{\partial}{\partial v_i} + \beta_3 n_e \frac{\partial}{\partial n_e}. \end{aligned} \tag{5}$$

Solutions of $QI = 0$ are obtained by solving the Lagrange subsidiary equations

$$\frac{dx}{\alpha_1 x} = \frac{dt}{\alpha_2 t} = \frac{dn_i}{\beta_1 n_i} = \frac{dv_i}{\beta_2 v_i} = \frac{dn_e}{\beta_3 n_e} = \frac{dI}{0}. \tag{6}$$

Using the theorem of Morgan (Morgan [7]), the invariants of the group G are the self-similar variables for the original partial differential equations,

$$\xi = \frac{x}{t}, \quad N_i(\xi) = t^2 n_i(x, t), \quad V_i(\xi) = v_i(x, t), \quad N_e(\xi) = t^2 n_e(x, t). \tag{7}$$

Substituting (7) into (2), we obtain the set of ordinary differential equations.

$$\begin{aligned} &-2N_i - \xi \frac{dN_i}{d\xi} + \frac{d(N_i V_i)}{d\xi} = 0, \\ &(V_i - \xi)\frac{dV_i}{d\xi} + \frac{1}{N_e}\frac{dN_e}{d\xi} = 0, \\ &\frac{d}{d\xi}\left(\frac{1}{N_e}\frac{dN_e}{d\xi}\right) = N_e - N_i. \end{aligned} \tag{8}$$

We note that the original set of equations (2) is invariant with respect to translational transformation in both space and time. Therefore all of the self-similar variables can be appropriately modified by replacing x and t by $x + x_0$ and $t + t_0$ where x_0 and t_0 are constants.

3. Infinitesimal transformation

The application of infinitesimal transformation groups to the solution of partial differential equations, was first discussed by Lie [3]. This technique is straight forward, although somewhat tedious, and allows one to determine the similarity variables without assuming the form of a group (i.e., linear, spiral, etc.) as was done in the previous section or in paper I. The details of the procedure appear in the book of Ames (Ames [2]), we shall therefore just present the basic equations which are found from the study of the plasma set (2).

We define the infinitesimal transformation as

$$\begin{aligned} \bar{x} &= x + \varepsilon X(x, t, u, v, w) + O(\varepsilon^2), \\ \bar{t} &= t + \varepsilon T(x, t, u, v, w) + O(\varepsilon^2), \\ \bar{u} &= u + \varepsilon U(x, t, u, v, w) + O(\varepsilon^2), \\ \bar{v} &= v + \varepsilon V(x, t, u, v, w) + O(\varepsilon^2), \\ \bar{w} &= w + \varepsilon W(x, t, u, v, w) + O(\varepsilon^2) \end{aligned} \tag{9}$$

where we've defined $u = n_i$, $v = v_i$ and $w = n_e$ to simplify notation.

The system of equations (2) is ACCI (Ames [2]) if the following sets of equations are satisfied. From (2a), we find

$$
\begin{aligned}
&U_t + uV_x + vU_x = 0,\\
&(U_u - T_t - T_x v - V_v + X_x)u + (U_v - T_x u - U_v)v - U = 0,\\
&(U_u - T_t - T_x v - U_u + X_x)v - V_u u - V - X_t = 0,\\
&U_v - T_x u - V_w uw - U_w vw = 0,\\
&U_w = 0,\\
&T_u u^2 + X_v u - T_v uv = 0, \qquad\qquad (10)\\
&X_u u - 2T_w uv = 0,\\
&-T_v v + X_v + T_u u - X_w vw + T_w v^2 w = 0,\\
&T_w u = 0,\\
&T_w v - X_w = 0,\\
&T_w = 0.
\end{aligned}
$$

From (2b), we find

$$
\begin{aligned}
&V_t w + V_x wv + W_x = 0,\\
&-V_u wv + W_u + V_u wv = 0,\\
&T_t wv + T_x wv^2 - V_u wu - X_t w + Vw - X_x wv + W_v = 0,\\
&-\frac{W}{w} - V_v + T_t + T_x v + V_w wv + W_w - X_x = 0, \qquad\qquad (11)\\
&V_w - T_x = 0,\\
&T_u = T_v = 0.
\end{aligned}
$$

From (2c), we find

$$
\begin{aligned}
&Uw^2 + Wuw - 2w^2 W + (W_w - 2X_x)(w^3 - w^2 u) + W_{xx} w = 0,\\
&2W_{xu} = 2W_{xv} = 2W_{uv} = W_{uu} = W_{vv} = W_u = W_v = 0,\\
&2W_{xw} w - X_{xx} w - 2W_x = 0,\\
&2W_{uw} w - 2W_u = 0, \qquad\qquad (12)\\
&2W_{vw} w - 2W_v = 0,\\
&\frac{W}{w} + W_{ww} w - W_w = 0.
\end{aligned}
$$

Many of the equations given in (10)–(12) are redundant. They can be reduced to the following

$$
\begin{aligned}
&U_t + uV_x + vU_x = 0,\\
&(U_u - T_t - V_v + X_x)u - U = 0,\\
&(X_x - T_t)v - V_u u - V + X_t = 0,\\
&V_t w + V_x wv + W_x = 0,\\
&T_t wv - V_u wu - X_t w + Vw - X_x wv = 0,\\
&-\frac{W}{w} - V_v + T_t + W_w - X_x = 0,\\
&Uw^2 + Wuw - 2w^2 W + W_w w^3 - W_x w^2 u - 2X_x w^3 + 2X_x w^2 u + W_{xx} w = 0,\\
&2W_{xw} w - X_{xx} w - 2W_x = 0,\\
&X_u = X_v = X_w = 0,\\
&T_x = T_u = T_v = T_w = 0,\\
&U_v = U_w = 0,\\
&V_w = 0;\\
&W_u = W_v = 0.
\end{aligned} \tag{13}
$$

The set of equations given in (13) can be solved to yield

$$X = \mu + \nu x, \qquad T = \rho + \nu t, \qquad U = -2\nu u, \qquad V = 0, \qquad W = -2\nu w, \tag{14}$$

where μ, ν and ρ are arbitrary constants.

The invariants of the transformation group

$$Q = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v} + W\frac{\partial}{\partial w} \tag{15}$$

can be found from the Lagrange subsidiary equations

$$\frac{dx}{X} = \frac{dt}{T} = \frac{du}{U} = \frac{dv}{V} = \frac{dw}{W} = \frac{dI}{0}. \tag{16}$$

The resulting invariants are

$$
\begin{aligned}
&\xi = \frac{\dfrac{\mu}{\nu} + x}{\dfrac{\rho}{\nu} + t},\\
&N_i(\xi) = \left(\frac{\rho}{\nu} + t\right)^2 n_i(x, t),\\
&V_i(\xi) = v_i(x, t),\\
&N_e(\xi) = \left(\frac{\rho}{\nu} + t\right)^2 n_e(x, t).
\end{aligned} \tag{17}
$$

We note that we have recovered the invariants predicted previously using an assumed linear group if we define $x_0 = \mu/\nu$ and $t_0 = \rho/\nu$. The resulting ordinary differential equations are given in (8).

We also note that if the constant ν were zero in (14), the analysis which follows would predict the traveling wave solution $N_i(\zeta) = n_i(x, t)$; $V_i(\zeta) = v_i(x, t)$ and $N_e(\zeta) = n_e(x, t)$ where $\zeta = x - (\mu/\rho)t$ for the set of equations (2).

4. Conclusions

In this paper, we have examined the self-similar nature of an important set of fluid equations which are used in plasma physics. Both the one-parameter transformation group and the infinitesimal transformation group gave identical results for the set of equations we treated. This is not, in general, the expected result (e.g., (Shen and Ames [4])). In addition, this infinitesimal transformation allowed for the recovery of the traveling wave solution.

Some comments can be made concerning solutions of the set of ordinary differential equations (8). There are no free parameters to now specify in order to do this. We however, can specify some terms based on physical arguments. For example, if N_iV_i = const (this implies that the ion flux n_iv_i decreases as t^{-2}), (see e.g., Widner, et al. [6]) we can integrate (8a) and find

$$N_i = \frac{K}{\xi^2} \quad \text{and} \quad V_i = \frac{\xi^2}{K'} \tag{18}$$

where K and K' are constants. From (8b), we find

$$N_e = N_{e0}\, \varepsilon^{(2\xi^2/3K') - \frac{1}{2}(\xi^2/K')^2}. \tag{19}$$

Unfortunately, (8c) is not identically satisfied.

In a future work, we shall investigate a well posed problem governed by (1) which will have a similarity solution. By well-posed, we mean that boundary conditions will be incorporated into the problem.

Acknowledgement

The authors wish to acknowledge Professor W. F. Ames and Dr. H. C. S. Hsuan for several discussions concerning the techniques applied in this paper.

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